The Line Integral

This integral is alternatively known as the **contour integral**. The reason is that the line integral involves integrating the projection of a vector field onto a specified **contour** C, e.g.,

$$\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Some important things to note:

* The integrand is a **scalar** function.

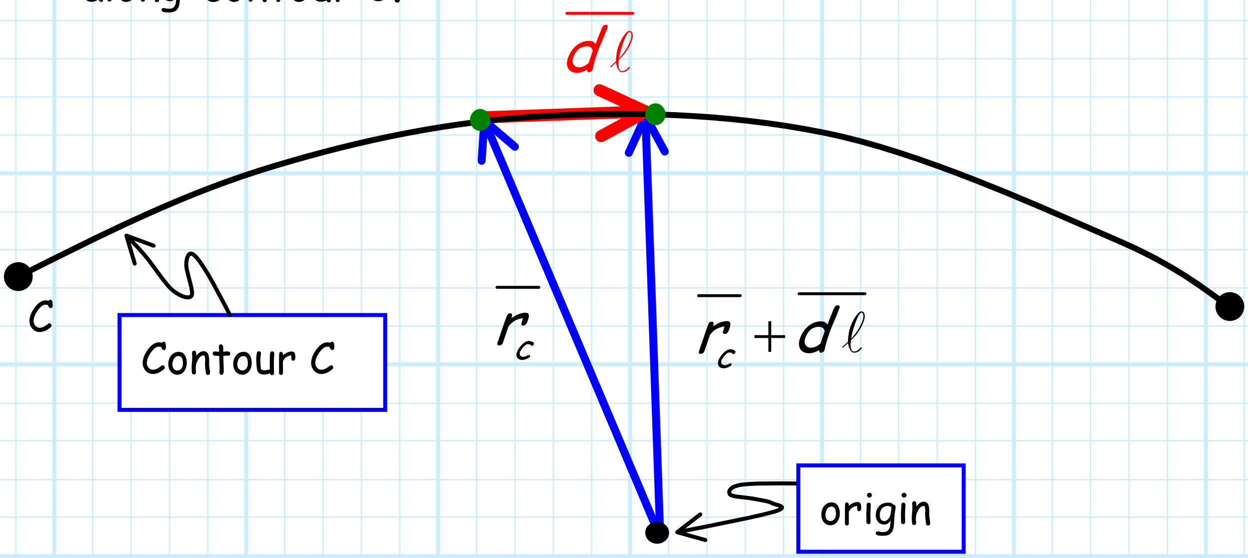
* The integration is over **one** dimension.

* The **contour** C is a line or curve through three-dimensional space.

* The position vector $\bar{r}_c$ denotes only those points that lie on contour C. Therefore, the value of this integral **only** depends on the value of vector field $\mathbf{A}(\bar{r})$ at the points along this contour.

Q: *What is the differential vector $\overline{d\ell}$, and how does it relate to contour C?*

A: The differential vector $\overline{d\ell}$ is the tiny **directed distance** formed when a point moves a small distance along contour C.



As a result, the differential line vector $\overline{d\ell}$ is **always tangential** to every point of the contour. In other words, the direction of $\overline{d\ell}$ always points "down" the contour.

Q: *So what does the scalar integrand* $\mathbf{A}(\overline{r}_c)\cdot\overline{d\ell}$ *mean? What is it that we are actually integrating?*

A: Essentially, the line integral integrates (i.e., "adds up") the values of a **scalar component** of vector field $\mathbf{A}(\overline{r})$ at **each and every point** along contour C. This scalar component of vector field $\mathbf{A}(\overline{r})$ is the projection of $\mathbf{A}(\overline{r}_c)$ onto the direction of the contour C.

First, I must point out that the notation $\mathbf{A}(\bar{r}_c)$ is **non-standard**. Typically, the vector field in the line integral is denoted simply as $\mathbf{A}(\bar{r})$. I use the notation $\mathbf{A}(\bar{r}_c)$ to emphasize that we are integrating the values of the vector field $\mathbf{A}(\bar{r})$ **only** at point that lie on contour C, and the points that lie on contour C are denoted as position vector $\bar{r}_c$.

In other words, the values of vector field $\mathbf{A}(\bar{r})$ at points that do not lie on the contour (which is just about all of them!) have no effect on the integration. The integral **only** depends on the value of the vector field as we move along contour C—we denote these values as $\mathbf{A}(\bar{r}_c)$.

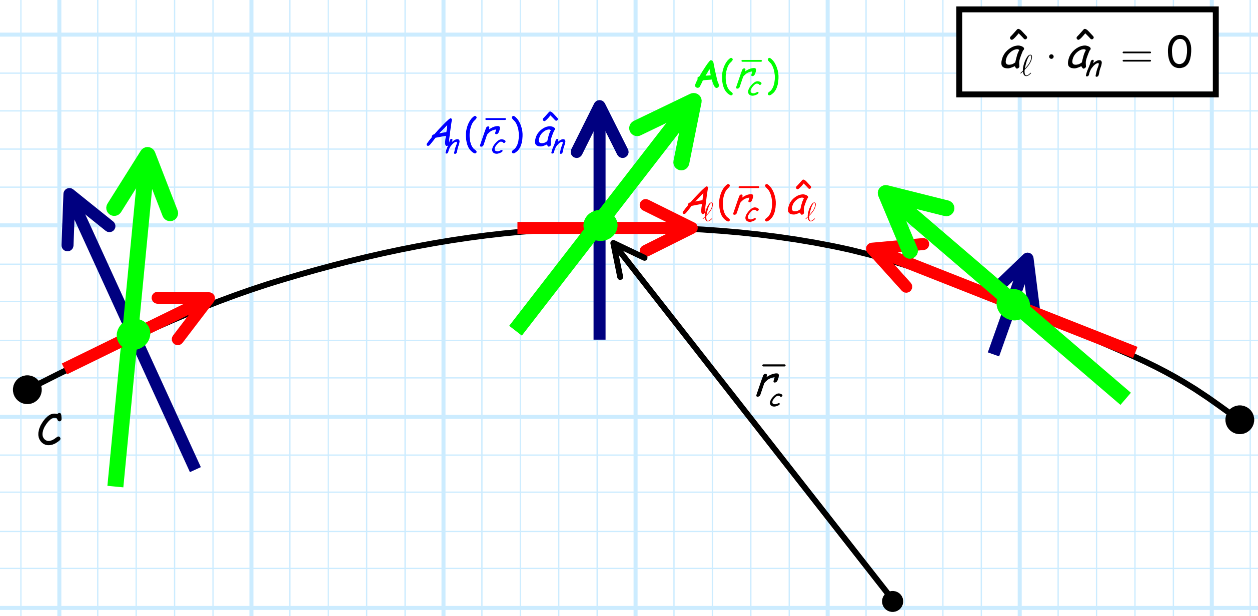
Moreover, the line integral depends on **only one scalar component** of $\mathbf{A}(\bar{r}_c)$!

Q: *On just what component of $\mathbf{A}(\bar{r}_c)$ **does** the integral depend?*

A: Look at the integrand $\mathbf{A}(\bar{r}_c)\cdot\overline{d\ell}$ —we see it involves the **dot** product! Thus, we find that the scalar integrand is simply the **scalar projection** of $\mathbf{A}(\bar{r}_c)$ onto the differential vector $\overline{d\ell}$. As a result, the integrand depends **only** the component of $\mathbf{A}(\bar{r}_c)$ that lies in the direction of $\overline{d\ell}$—and $\overline{d\ell}$ **always** points in the direction of the contour C!

To help see this, first note that $\mathbf{A}(\bar{r}_c)$, the value of the vector field along the contour, can be written in terms of a vector component **tangential** to the contour (i.e, $A_\ell(\bar{r}_c)\,\hat{a}_\ell$), and a vector component that is **normal** (i.e., orthogonal) to the contour (i.e, $A_n(\bar{r}_c)\,\hat{a}_n$):

$$\mathbf{A}(\bar{r}_c) = A_\ell(\bar{r}_c)\,\hat{a}_\ell + A_n(\bar{r}_c)\,\hat{a}_n$$



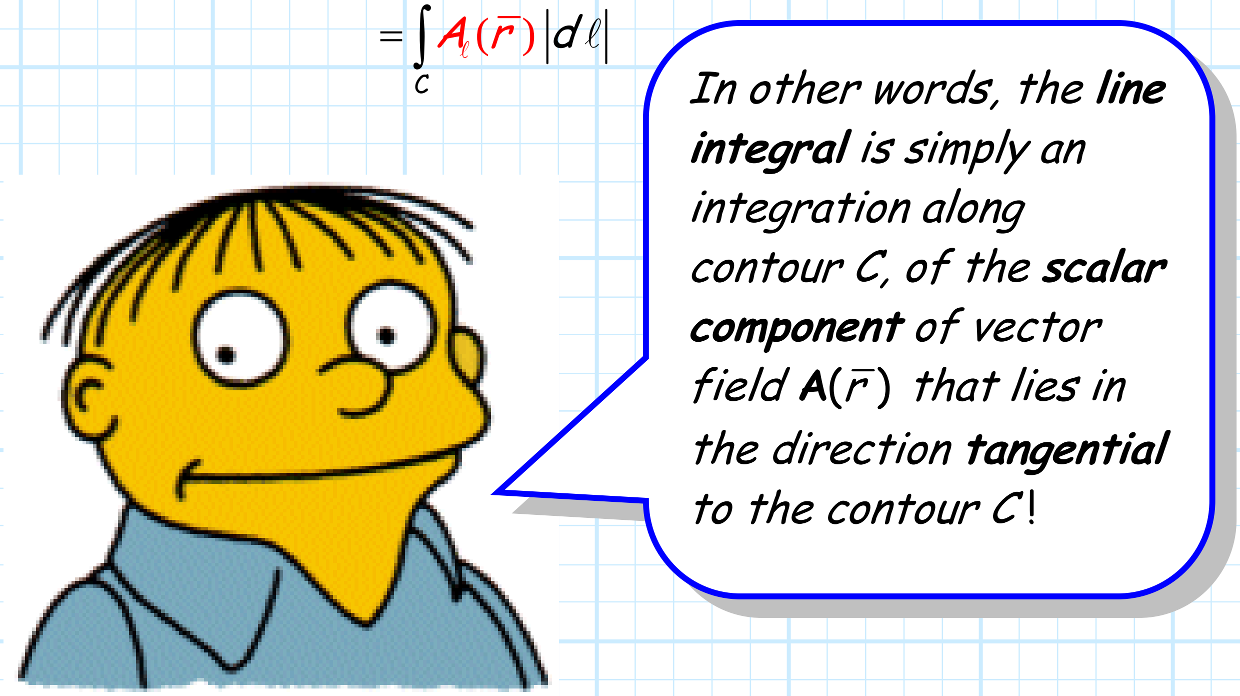
We likewise note that the differential line vector $\overline{d\ell}$, like **any** and all vectors, can be written in terms of its magnitude ($|d\ell|$) and direction ($\hat{a}_\ell$) as:

$$\overline{d\ell} = \hat{a}_\ell\,|d\ell|$$

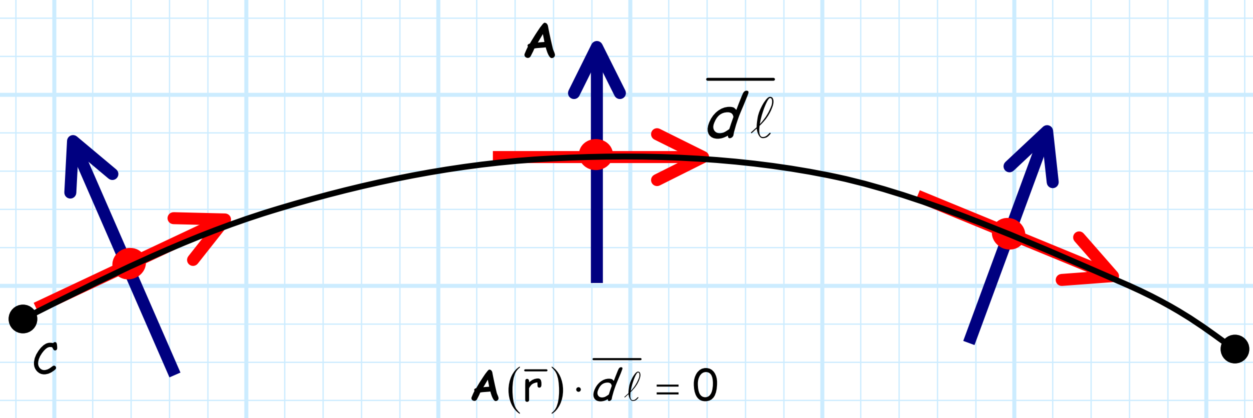
For example, for $\overline{d\phi} = \rho\, d\phi\, \hat{a}_\phi$, we can say $|d\ell| = \rho\, d\phi$ and $\hat{a}_\ell = \hat{a}_\phi$.

As a result we can write:

$$\begin{aligned}\int_C \mathbf{A}(\bar{\mathbf{r}}_c)\cdot\overline{d\ell} &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\overline{d\ell} \\ &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\,\hat{a}_\ell\,|d\ell| \\ &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell\cdot\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\cdot\hat{a}_\ell\right]|d\ell| \\ &= \int_C A_\ell(\bar{r})\,|d\ell|\end{aligned}$$



Note if vector field $\mathbf{A}(\bar{\mathbf{r}})$ is **orthogonal** to the contour at every point, then the resulting line integral will be **zero**.



Although C represents **any** contour, no matter how **complex** or **convoluted**, we will study only **basic** contours. In other words, $\overline{d\ell}$ will correspond to one of the differential line vectors we have **previously** determined for Cartesian, cylindrical, and spherical coordinate systems.